Contour Map (Elevation Map) of Mt. St. Helens from 1979:

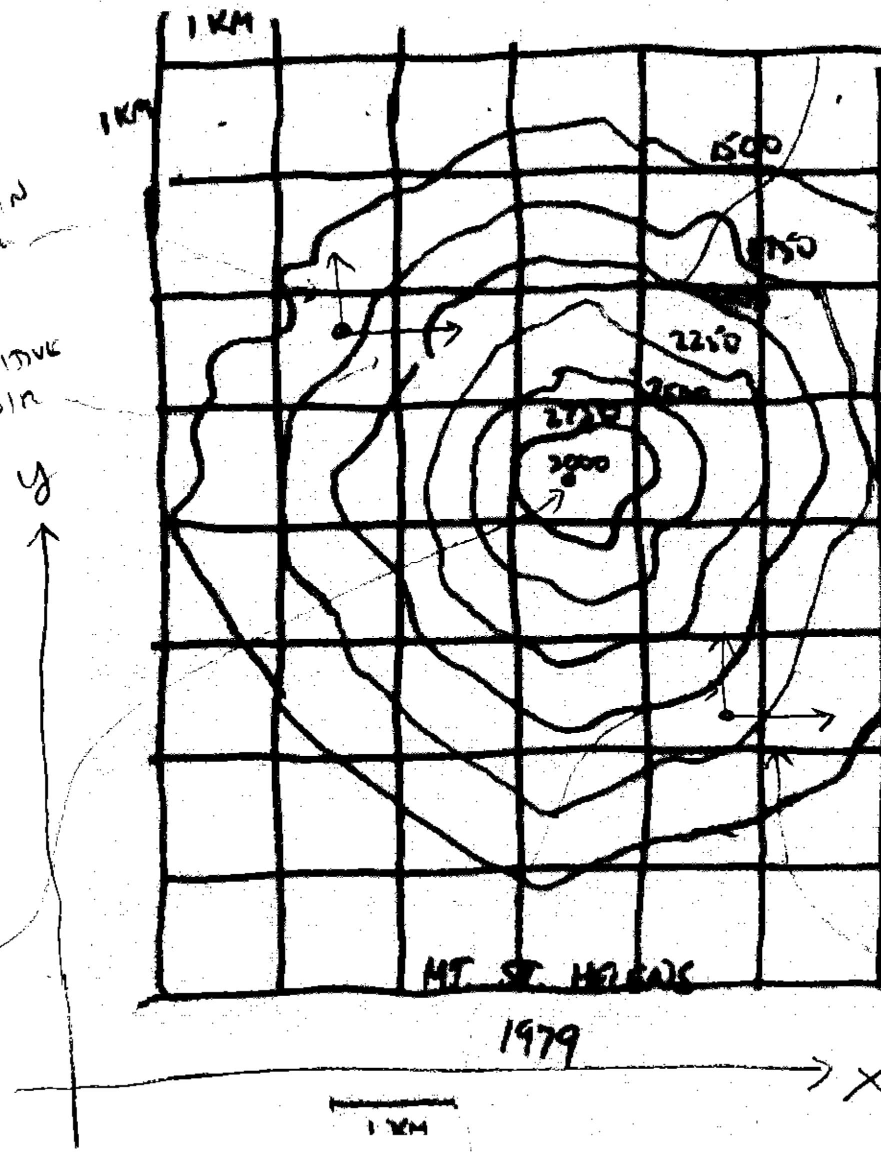
## 14.1/14.3 Visualizing Surfaces and Partial Derivatives

The basic tool for visualizing surfaces is **traces**. When $z = f(x, y)$, we look at traces for fixed $z$-values (heights) first. We call these traces **level curves**.

A collection of level curves is called a **contour map** (or **elevation map**).

SLOPE NEG. IN y-DIR

SLOPE POSITIVE IN X-DIR




LOCAL MAX (SLOPE ZERO IN ALL-DIRECTIONS HERE)

SLOPE POSITIVE IN y-DIRECTION HERE

SLOPE NEGATIVE IN X-DIRECTION HERE

*Example*: Draw a contour map for

$$z = f(x, y) = y - x$$

$\boxed{z=0}$

$0 = y - x \Rightarrow y = x$

$\boxed{z=1}$

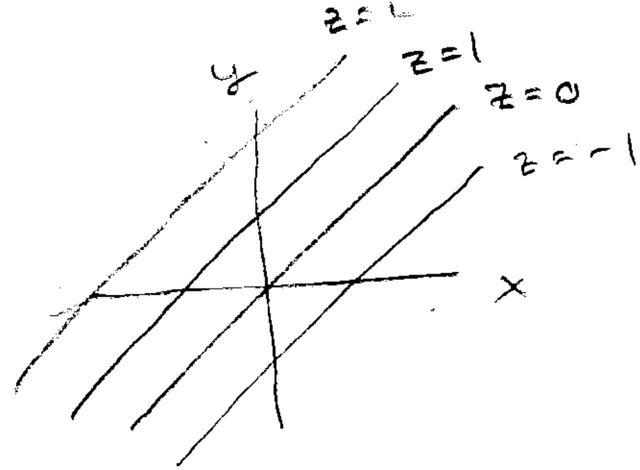
$1 = y - x \Rightarrow y = x + 1$

$\boxed{z=2}$

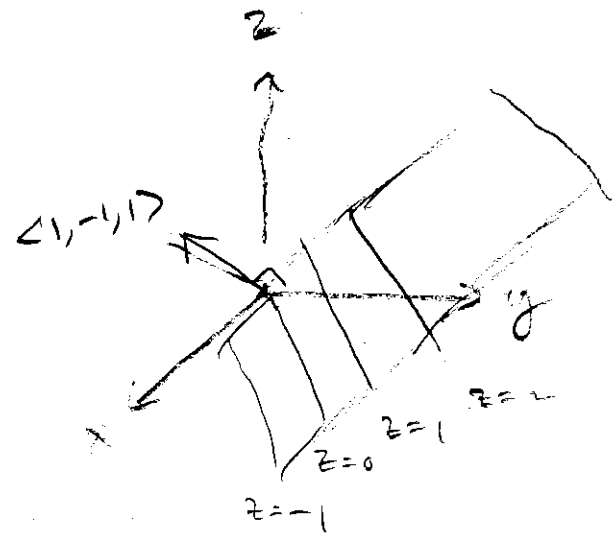
$2 = y - x \Rightarrow y = x + 2$

$\boxed{z=k}$

$k = y - x \Rightarrow y = x + k$




IT IS A PLANE!!




$z = y - x \Leftrightarrow x - y + z = 0$

$\langle 1, -1, 1 \rangle = \vec{n}$

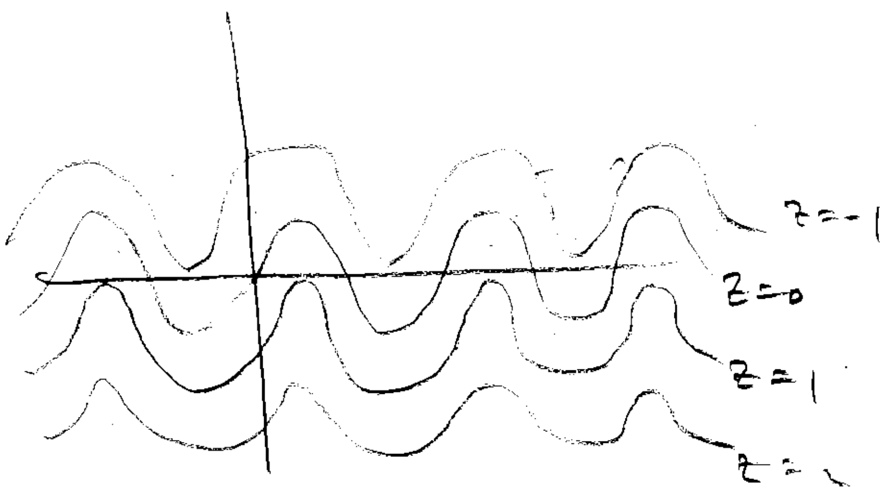
*Example*: Draw a contour map for

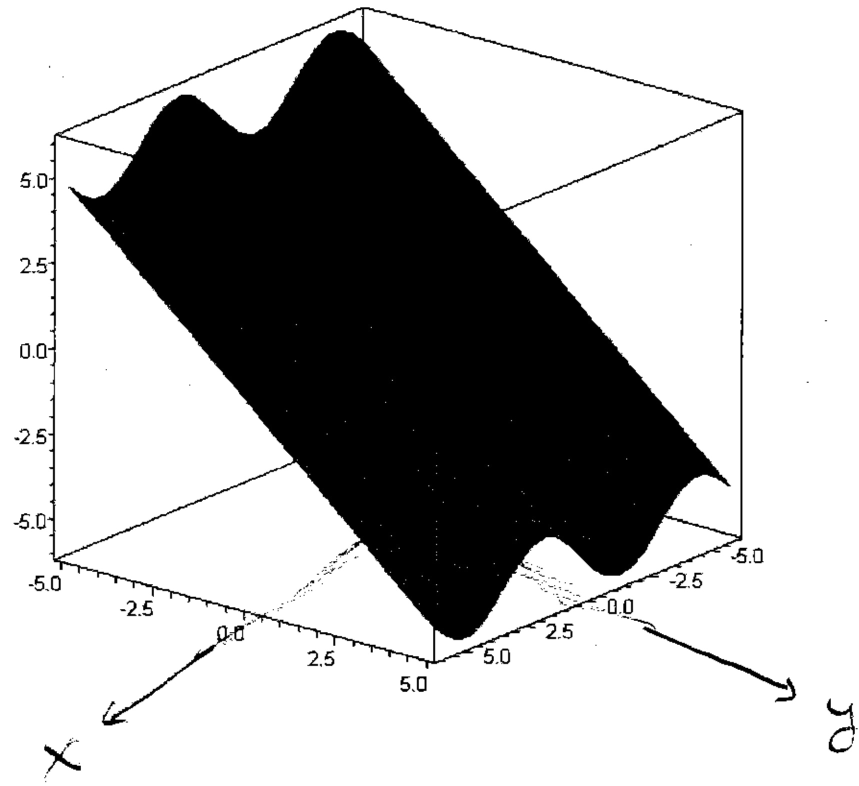
$$z = \sin(x) - y$$

$\boxed{z = k}$

$k = \sin(x) - y$

$y = \sin(x) - k$



*Example*: Draw a contour map for

$$z = f(x,y) = \frac{1}{1 + x^2 + y^2}$$

(use z = 1/10, 2/10, … , 9/10, 10/10)

$$k = \frac{1}{1+x^2+y^2}$$

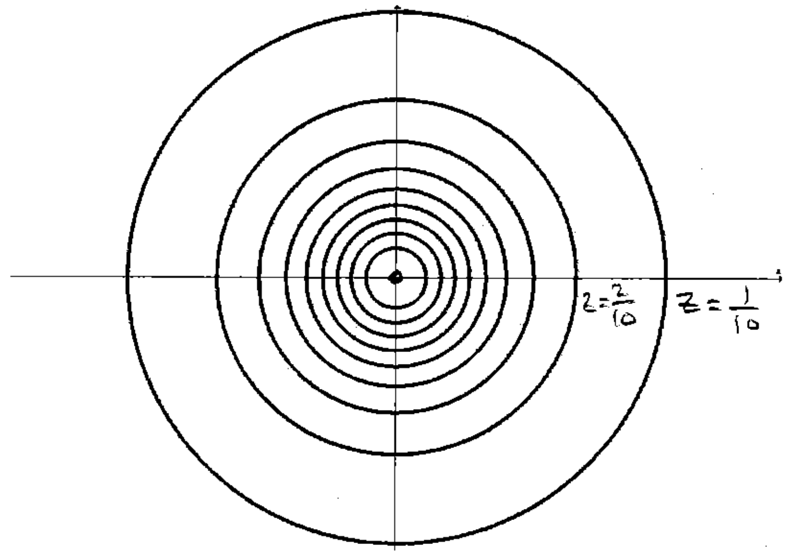
$$k(1+x^2+y^2) = 1$$

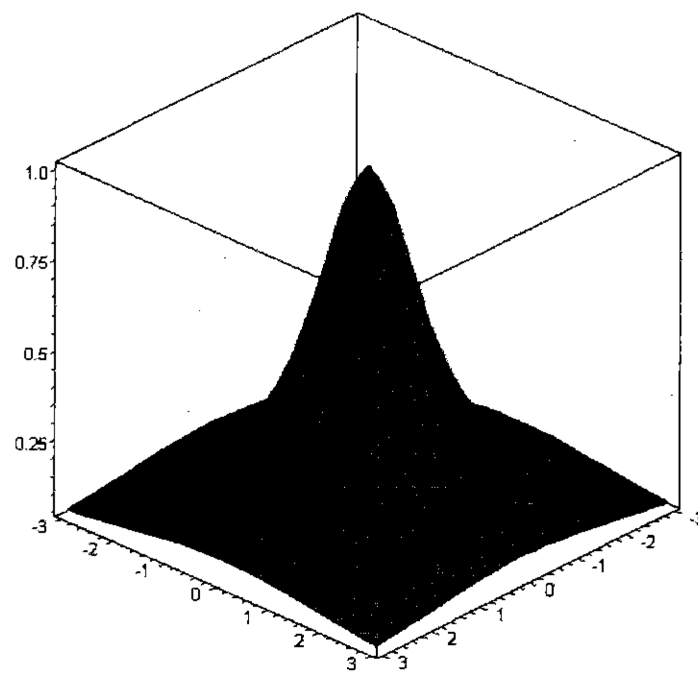
$$1+x^2+y^2 = 1/k$$

$$x^2+y^2 = \underbrace{1/k - 1}$$

CIRCLE OF RADIUS $\sqrt{\frac{1}{k} - 1}$

ONLY MAKES SENSE FOR $0 < k \leq 1$



A question that asks “find the ***domain***” is asking if you know your functions well enough to understand when they are defined and not defined.

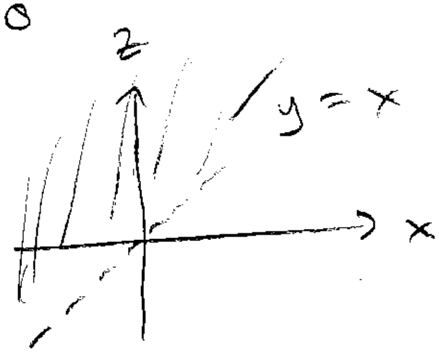
| *Appears in Function* | *Restriction* |
|---|---|
| $\sqrt{BLAH}$ | BLAH $\geq$ 0 |
| STUFF/BLAH | BLAH $\neq$ 0 |
| ln(BLAH) | BLAH > 0 |
| $\sin^{-1}$(BLAH) | -1 $\leq$ BLAH $\leq$ 1 |
| and other trig… | |

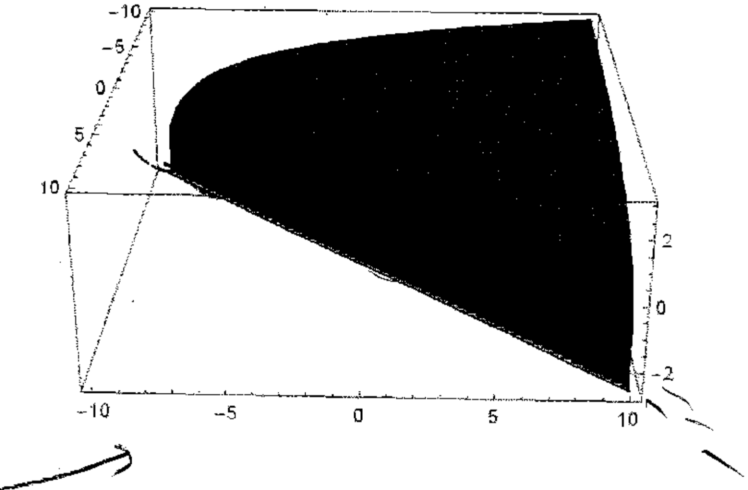
*Examples:* Sketch the domain of

(1) $f(x,y) = \ln(y - x)$

$y - x > 0$

$y > x$

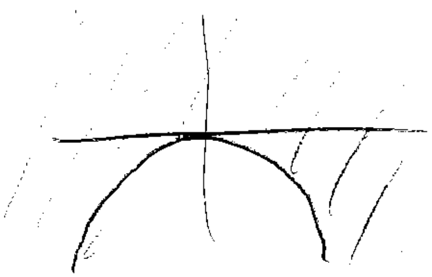


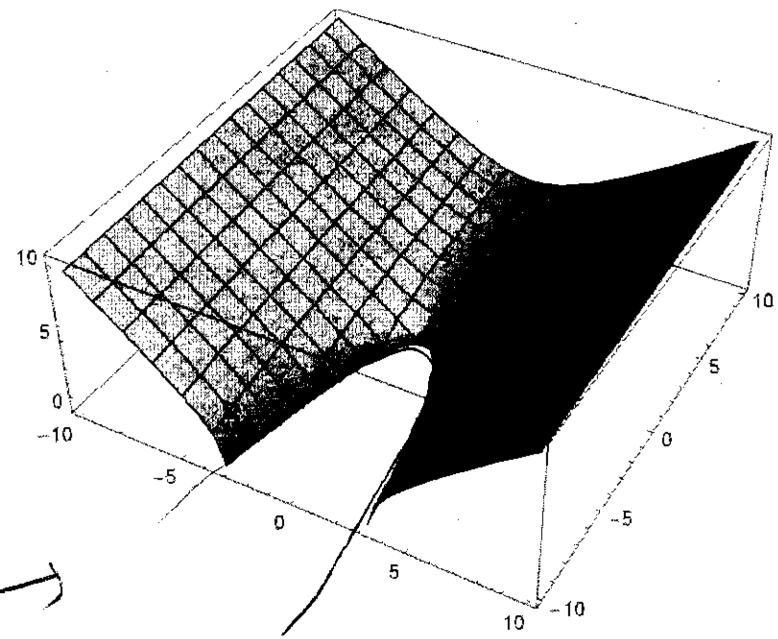




(2) $g(x,y) = \sqrt{y + x^2}$

$y + x^2 \geq 0$

$y \geq -x^2$



## 14.3 Partial Derivatives

Goal: Get the slope in two different directions on a surface.

Recall the key def'n for all calculus

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Today we define:

FIXED CONSTANT

$$\frac{\partial z}{\partial x} = f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

FIXED CONSTANT

$$\frac{\partial z}{\partial y} = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

*Motivation*: Consider

$$f(x,y) = x^2y + 5x^3 + y^2$$

Find

a. $\frac{d}{dx}[f(x,2)] = \frac{d}{dx}[x^2(2) + 5x^3 + (2)^2]$

$= 2x\,(2) + 5 \cdot 3x^2 + 0 = 4x + 15x^2$

b. $\frac{d}{dx}[f(x,3)] = \frac{d}{dx}[x^2(3) + 5x^3 + (3)^2]$

$= 2x\,(3) + 5 \cdot 3x^2 + 0 = 6x + 15x^2$

c. $\frac{d}{dx}[f(x,c)] = \frac{d}{dx}[x^2(c) + 5x^3 + (c)^2]$

$= 2x\,(c) + 5 \cdot 3x^2 + 0 = 2xc + 15x^2$

$$\boxed{\frac{\partial z}{\partial x} = 2xy + 15x^2}$$

WHAT IS $\frac{\partial z}{\partial y}$?

$f_y(x,y) = x^2 \cdot 1 + 0 + 2y$

$$\boxed{\frac{\partial z}{\partial y} = x^2 + 2y}$$

Example:

$$f(x,y) = \overset{\text{COEF.}}{x^3 y} + \underbrace{x^5}_{F}\underbrace{e^{xy^2}}_{S} + \overset{\text{CONSTANT (NO } x\text{'S)}}{\ln(y)}$$

$$\frac{\partial z}{\partial x} = 3x^2 y + \underbrace{5x^4}_{F'}\underbrace{e^{xy^2}}_{S} + \underbrace{x^5}_{F}\underbrace{y^2 e^{xy^2}}_{S'} + 0$$

$$\boxed{\frac{\partial z}{\partial x} = 3x^2 y + 5x^4 y^2 e^{xy^2}}$$

$$f(x,y) = \overset{\text{COEF.}}{x^3} y + \overset{\text{COEF}}{x^5} e^{xy^2} + \ln(y)$$

$$\frac{\partial z}{\partial y} = x^3(1) + x^5 \cdot e^{xy^2} \cdot 2xy + \frac{1}{y}$$

$$\boxed{\frac{\partial z}{\partial y} = x^3 + 2x^6 y e^{xy^2} + \frac{1}{y}}$$

Example:

$$g(x,y) = \cos(x^3 + y^4)$$

$$g_x(x,y) = -\sin(x^3+y^4)\,(3x^2+0)$$

$$\boxed{g_x(x,y) = -3x^2 \sin(x^3+y^4)}$$

$$g_y(x,y) = -\sin(x^3+y^4) \cdot (0+4y^3)$$

$$\boxed{g_y(x,y) = -4y^3 \sin(x^3+y^4)}$$

## Important Note on Variables

A variable can be treated as:

1. A constant
2. An independent variable (input)
3. A dependent variable (output),

*Examples*:

a) **One variable function of *x***:

OUTPUT

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

INPUT

b) **Related rates**:

At time *t* assume a particle is moving along the path $y = x^2$ $\Leftrightarrow$ $y(t) = (x(t))^2$

$x = x(t)$

$y = y(t)$

$$\frac{dy}{dt} = 2x\frac{dx}{dt}$$

c) **Implicit functions**: $x^2 + y^2 = 1$ $\Leftrightarrow$ $x^2 + (y(x))^2 = 1$

$y = y(x)$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$2x + 2(y(x))\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{2y}$

## d) Multivariable:

ALL CONSTANT  COEF.

$$z = x^2 + y^3e^{6y} - 5xy^4$$

CONSTANT  COEF.

$z = x^2 + y^3 e^{6y} - 5xy^4$

OUTPUT

$$\frac{\partial z}{\partial x} = 2x + 0 - 5(1)y^4 = \boxed{2x - 5y^4}$$

INPUT

$$\frac{\partial z}{\partial y} = 0 + \underbrace{y^3}_{F}\underbrace{e^{6y}\cdot 6}_{S'} + \underbrace{3y^2}_{F'}\cdot\underbrace{e^{6y}}_{S} - 5x(4y^3) = \boxed{6y^3e^{6y} + 3y^2e^{6y} - 20xy^3}$$

## e) Multivariable Implicit:

CONSTANT

$$x^2 + y^2 - z^2 = 1$$

$x^2 + y^2 - (z(x))^2 = 1$

$$\frac{\partial z}{\partial x} \quad 2x + 0 - 2z\frac{\partial z}{\partial x} = 0 \Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{-2x}{-2z} = \frac{x}{z}}$$

$$\frac{\partial z}{\partial y} =$$

$x^2 + y^2 - z^2 = 1$

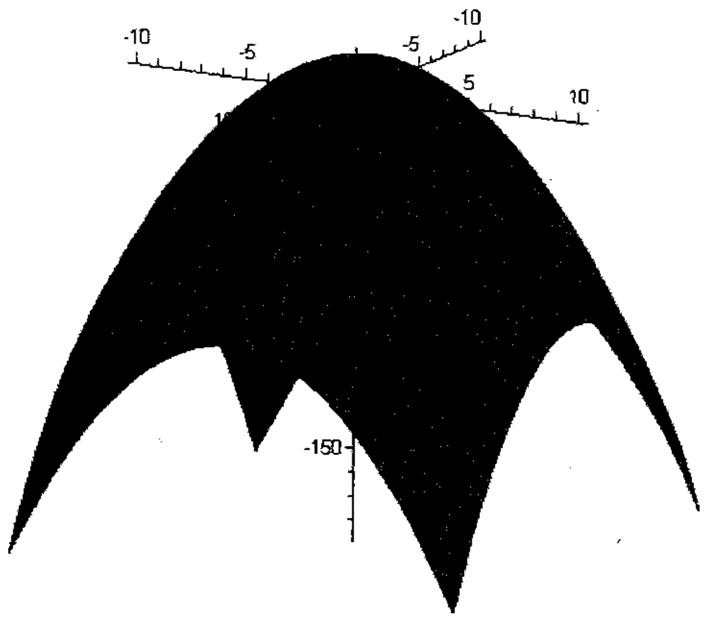
CONSTANT  $z(y)$

$0 + 2y - 2z\frac{\partial z}{\partial y} = 0$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-2y}{-2z} = y/z}$$

*Graphical Interpretations*:

Pretend you are skiing on the surface
$z = f(x, y) = 15 - x^2 - y^2$




$f_x(x,y) = -2x$

$f_y(x,y) = -2y$

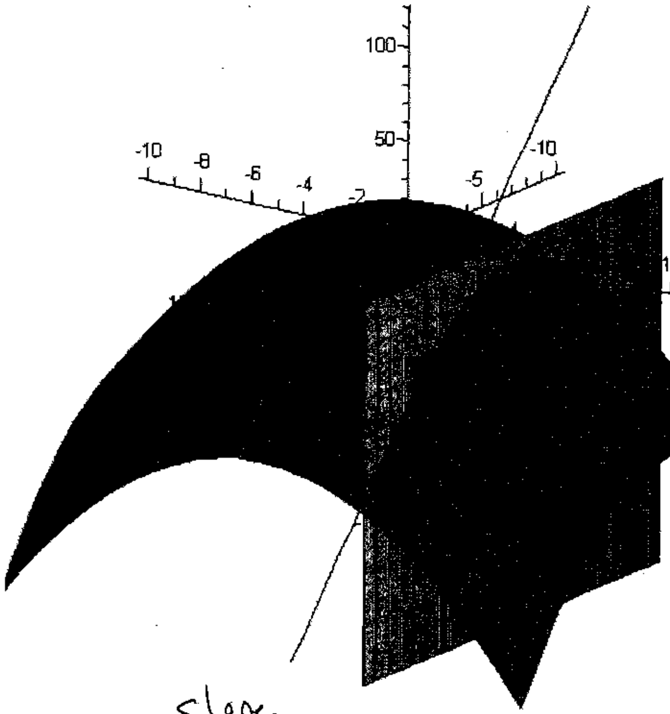
$f(4,7) = 15-(4)^2-(7)^2 = 15-16-49$

$\boxed{f(4,7) = -50}$ HEIGHT

$f_x(4,7) = -8$ "SLOPE IN X-DIRECTION"

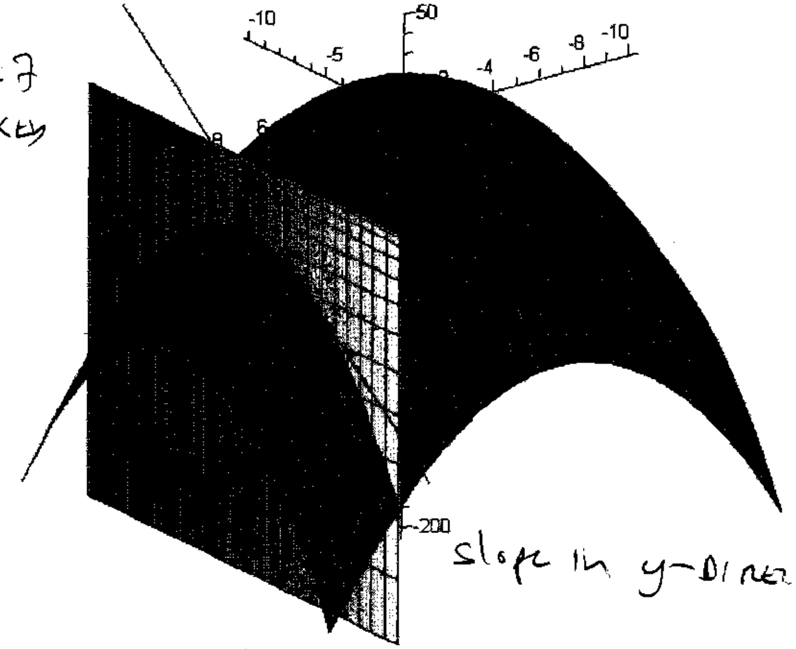
$f_y(4,7) = -14$ "SLOPE IN Y-DIRECTION"

$y = 4$
FIXED

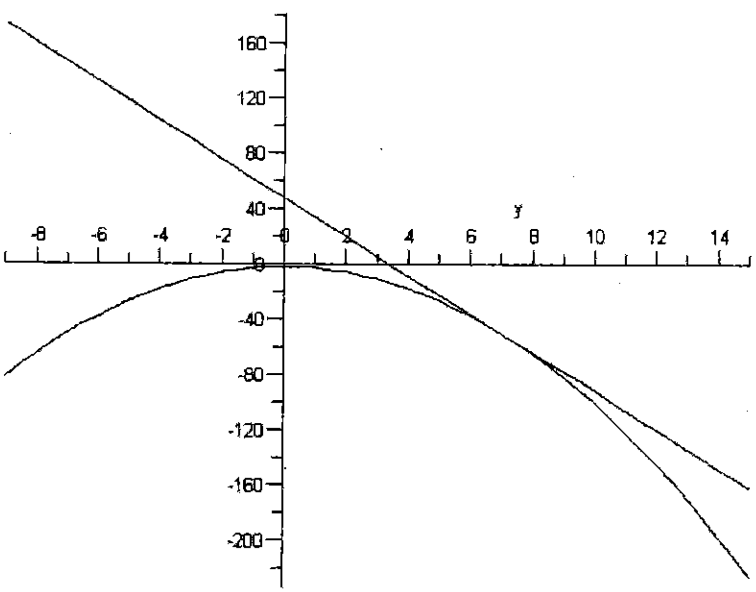


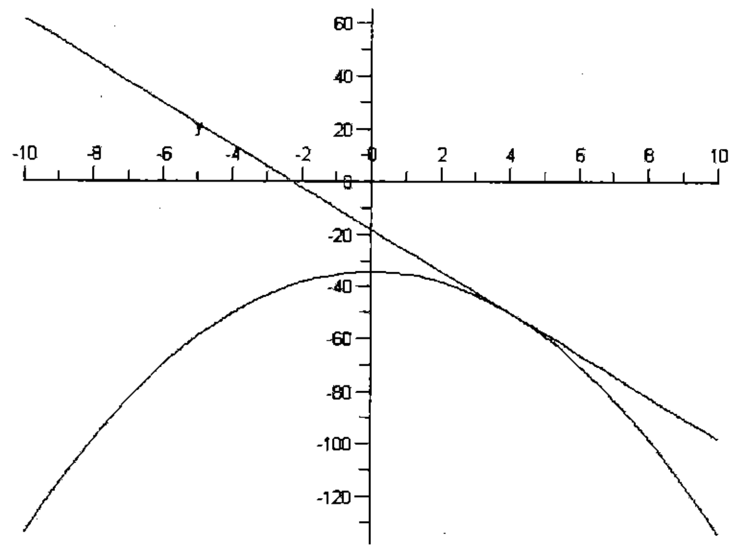
slope
in x-DIRECTION

$x = 7$
FIXED



slope in y-DIRECTION





**Second Partial Derivatives**

Concavity in *x*-direction:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{xx}(x, y)$$

Concavity in *y*-direction:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f_{yy}(x, y)$$

Mixed Partials:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{xy}(x, y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{yx}(x, y)$$

*Example*: Find all second partials for

$z = f(x, y) = x^4 + 3x^2y^3 + y^5$

$f_x = 4x^3 + 6xy^3$

$f_y = 9x^2y^2 + 5y^4$

$f_{xx} = 12x^2 + 6y^3$

$f_{xy} = 18xy^2 = f_{yx}$

$f_{yy} = 18x^2y + 20y^3$

ALWAYS THE SAME!

(IF BOTH DEFINED AND CONTINUOUS)

CLAIRAUT'S THEOREM

(Writer of 1st systematic discussion on 3D Calculus on surfaces)